Completeness Property of $\mathbb{R}$

every nonempty subset of $\mathbb{R}$ ~~which~~ that has an upper bound also has a supremum in $\mathbb{R}$ (lower) (infimum)

~~Re theorem~~ Que :- let $S \subseteq \mathbb{R}$, $S \neq \phi$ which is bounded above then prove that $\sup(a+S) = a + \sup S$, $a \in \mathbb{R}$

$a + S = \{a + x \mid x \in S\}$

if S is bounded above then $-S$ is bounded below

Sol^n :- $\because$ S is bounded above

$\Rightarrow$ $a+S$ is bounded above

therefore by completeness property of $\mathbb{R}$

$\sup(S)$ and $\sup(a+S)$ exist

let $u = \sup(S)$

$\Rightarrow x \leq u \quad \forall x \in S$

$\Rightarrow a+x \leq a+u \quad \forall x \in S$

$\Rightarrow a+u$ is an upper bound of $a+S$

$\Rightarrow \sup(a+S) \leq a+u$ — (I)

Now let v be any upper bound of $a+S$

$\Rightarrow a+x \leq v \quad \forall x \in S$

$\Rightarrow x \leq v-a \quad \forall x \in S$

$\Rightarrow v-a$ is an upper bound of S

$\Rightarrow u \leq v-a \quad \{\because u = \sup(S)\}$

$\Rightarrow u+a \leq v \quad \{\because v$ is upper bound of $a+S$

$\Rightarrow u+a \leq \sup(a+S)$ — (II) $\{\because$ ~~it at S~~

$\Rightarrow$ from (I) & (II)

$\Rightarrow u+a = \sup(a+S)$

$\Rightarrow a + \sup(S) = \sup(a+S)$

Qu Let A & B are two nonempty subsets of R such that $a \le b$ $\forall a \in A$ & $b \in B$ then $\sup A \le \inf B$.

Proof :- $\because$ $a \le b$ $\forall a \in A$ & $b \in B$

$\Rightarrow$ b is upper bound of A; $\forall b \in B$

$\Rightarrow$ $\sup A \le b$ $\forall b \in B$

$\Rightarrow$ $\sup A$ is lower bound of B

$\Rightarrow$ $\sup(A) \le \inf(B)$

Ques :- Let $S_0 \subseteq S \subseteq \mathbb{R}$ & S is bounded set then prove that $\inf S \le \inf S_0 \le \sup S_0 \le \sup S$

Soln :- $\because$ S is bounded & S_0 is subset of S

$\Rightarrow$ S_0 is also bounded.

$\therefore$ by completeness property of $\mathbb{R}$

Both S & S_0 have their supremum and infimum.

first we claim :- $\sup S_0 \le \sup S$

let $u = \sup(S)$ & $u_0 = \sup(S_0)$

Now $x \le u$ $\forall x \in S$

as $S_0 \subseteq S$

$x_0 \le u$ $\forall x_0 \in S_0$

$\Rightarrow$ x_0 ... u is upper bound of S_0

$\Rightarrow$ u is ... $\Rightarrow$ $\sup S_0 \le \sup S$ — (i)

$\Rightarrow \sup S_0 \le u$

Now we claim $\inf S \le \inf S_0$

let $y = \inf S$ and $y_0 = \inf S_0$

$\Rightarrow$ $y \le x$ $\forall x \in S$

& $S_0 \subseteq S$

$y \le x_0$ $\forall x_0 \in S_0$ $\Rightarrow$ y is lower bound of S_0

$\Rightarrow$ $y \le \inf S_0$ — (ii)

$\Rightarrow$ $y \le \inf S \le \sup S$ — (iii)

and by definition, $\inf S_0 \le \sup S_0 \le u$

$\Rightarrow$ $y \le \inf S_0$

$\Rightarrow$ $\inf S \le \inf S_0 \le \sup S_0 \le \sup S$

(11)

Qn :- Let S be non-empty subset of $\mathbb{R}$ which is bounded below then prove that $\inf S = -\sup(-S)$

Solⁿ :-

$\because$ $S(\neq \phi) \subseteq \mathbb{R}$ which is bounded below

$\Rightarrow$ $\exists$ $K \in \mathbb{R}$ such that

$x \geq K \; \forall x \in S$

$\Rightarrow$ $-x \leq -K \; \forall x \in S$

$\Rightarrow$ $-K$ is upper bound of $(-S)$

$\Rightarrow$ $-S$ is bounded above

$\Rightarrow$ By completeness property of $\mathbb{R}$

$\inf(S)$ and $\sup(-S)$ exist

Let $l = \inf(S)$ & $u = \sup(-S)$

To prove :- $l = -u$

as $l = \inf S$

$\Rightarrow x \geq l \; \forall x \in S$

$\Rightarrow -x \leq -l \; \forall x \in S$

$\Rightarrow -l$ is upper bound of $(-S)$

$\Rightarrow \sup(-S) \leq -l$

$\Rightarrow u \leq -l$ — (I)

as $u = \sup(-S)$

$\Rightarrow u \geq -x \; \forall x \in S$

$\Rightarrow x \geq -u \; \forall x \in S$

$\Rightarrow -u$ is lower bound of S

$\Rightarrow \inf S \geq -u$

$\Rightarrow l \geq -u$ — ~~(II)~~

$\Rightarrow -l \leq u$ — (II)

from (I) & (II)

$l = -u$

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Archimedean property :-

if $x \in \mathbb{R}$, then $\exists\, n_x \in \mathbb{N}$ such that $x \leq n_x$

Proof :- suppose that $n_x \leq x \;\forall\, n_x \in \mathbb{N}$

$\Rightarrow$ x is the upper bound of $\mathbb{N}$

$\Rightarrow$ By completeness property $\mathbb{N}$ has supremum in $\mathbb{R}$ say u

as $u-1 < u$

$\Rightarrow$ $u-1$ is not upper bound of $\mathbb{N}$

$\Rightarrow$ $\exists\, m \in \mathbb{N}$ such that $m > u-1$

$\Rightarrow$ $u < m+1$ & $m+1 \in \mathbb{N}$.

which is a contradiction

$\Rightarrow$ $\exists\, n_x \in \mathbb{N}$ such that $x < n_x$.

Corollary :- let $S = \{\frac{1}{n};\, n \in \mathbb{N}\}$, then $\inf S = 0$

Soln :- Here $S \neq \phi$ & bounded below by 0

because $\frac{1}{n} > 0 \Rightarrow n > 0$

$\Rightarrow$ By completeness property

$\inf S$ exist

let $\omega = \inf S$ & $\omega \geq 0$

for every $\varepsilon > 0$, By archimedean property

$\exists\, n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < n \Rightarrow \frac{1}{n} < \varepsilon$

Now $0 \leq \omega \leq \frac{1}{n} \Rightarrow 0 \leq \omega < \varepsilon$ $\{\because \omega$ is $\inf S\}$

$\Rightarrow \omega = 0$ $\{\because$ if $a \in \mathbb{R}$ such that $0 \leq a < \varepsilon$ for $\varepsilon > 0$ then $a = 0\}$.

Corollary :- If $t > 0$, there exist $n_t \in \mathbb{N}$ such that $0 < \frac{1}{n_t} < t$.

Proof :- Consider $S = \{\frac{1}{n} ; n \in \mathbb{N}\}$

$\Rightarrow \inf S = 0$

As $t > 0$ then t is not a lower bound of S

$\Rightarrow \exists n_t \in \mathbb{N}$ such that $\frac{1}{n_t} < t$

$\Rightarrow 0 < \frac{1}{n_t} < t$.

Corollary :- If $y > 0$ $\exists n_y \in \mathbb{N}$ such that $n_y - 1 \leq y < n_y$

Proof :- Consider

$E_y = \{m \in \mathbb{N},\ y < m\}$

$\Rightarrow$ By Archimedean property $E_y \neq \phi$

Now By well ordering property E_y has least element say n_y {every nonempty subset of $\mathbb{N}$ has least element}

$\Rightarrow n_y - 1 \notin E_y$

$\Rightarrow n_y \leq y$ & $y < n_y$ {$\because n_y$ is ~~ele~~ least element of E_y}

$\Rightarrow n_y - 1 \leq y < n_y$.

2.4 Q(1) Show that $\sup\{1 - \frac{1}{n} ; n \in \mathbb{N}\} = 1$

Sol^n here $S \neq \phi$

$S = \{0, \frac{1}{2}, \frac{2}{3}, \frac{3}{4}, \ldots\}$

element of S are increasing and it will reach to 1 at $n \to \infty$

S is bounded above by 1

let $\alpha = \sup S$

$\Rightarrow \alpha \leq 1$ —(1)

for $\varepsilon > 0$, and applying archimedean property

$\Rightarrow \exists n \in \mathbb{N}$ such that $\frac{1}{\varepsilon} < n \Rightarrow \frac{1}{n} < \varepsilon$

$\Rightarrow \frac{1}{n} < \varepsilon$

$\Rightarrow 1-\frac{1}{n} > 1-\varepsilon$ —(ii)

$\Rightarrow 1-\varepsilon < 1-\frac{1}{n} < x \leq 1$ {from (i) & (ii)}

$\Rightarrow 1-\varepsilon < x \leq 1$

$\Rightarrow 1 \leq x \leq 1$

$\Rightarrow x = 1$

Qu. (2) If $S = \{\frac{1}{n} - \frac{1}{m} ; n, m \in \mathbb{N}\}$ find sup S & inf S

Soln: Here $S \neq \phi$

{ $\because 0 \leq \frac{1}{m} \leq 1$ $\Rightarrow -1 \leq -\frac{1}{m} < 0$

as $0 \leq \frac{1}{n} \leq 1$ & $-1 \leq -\frac{1}{m} < 0$.

$\Rightarrow -1 \leq \frac{1}{n} - \frac{1}{m} \leq 1$

$\Rightarrow -1 \leq x \leq 1 \quad \forall x \in S$

let $p = \sup S$ and $q = \inf S$

$\Rightarrow p \leq 1$ & ~~$q \leq -1$~~ $q \geq -1$

for $\varepsilon > 0$, applying the archimedean property

$\Rightarrow \exists n \in \mathbb{N}$ such that

$\frac{1}{\varepsilon} < n \Rightarrow \frac{1}{n} < \varepsilon$ $\quad \frac{1}{\varepsilon} < m \Rightarrow \frac{-1}{m} > -\varepsilon$

$\Rightarrow \exists m \in \mathbb{N}$ such that

$\Rightarrow -\varepsilon < \frac{1}{n} - \frac{1}{m} < \varepsilon$

$\Rightarrow -\varepsilon < -1 \leq q < \frac{1}{n} - \frac{1}{m} < \varepsilon$

& $-\varepsilon < \frac{1}{n} - \frac{1}{m} < p \leq 1 < \varepsilon$

~~as $-1 \leq q$ & $p \leq 1$~~

$\Rightarrow q = -1$ & $p = 1$

$-\varepsilon < -1 \leq q$

$\Rightarrow -q \leq 1 < \varepsilon$

$\Rightarrow 0 \leq 1+q < \varepsilon$

$\Rightarrow 1+q = 0$

$\Rightarrow q = -1$

IIly $p = 1$

Ques (3) $S (\neq \Phi) \subseteq \mathbb{R}$

$u \in \mathbb{R}$

$n \in \mathbb{N}$; $u - 1/n$ is not upper bound of S

$n \in \mathbb{N}$; $u + 1/n$ is a upper bound of S

To prove: $u = \sup S$

let $x = \sup S \quad \{x \in S\}$

$\Rightarrow u + 1/n \geq x$

$\Rightarrow x \leq u \leq u + 1/n$

$\Rightarrow x \leq u$

$\Rightarrow \sup S \leq u$ — (i)

let $t \in S$

$\Rightarrow u - 1/n < t \leq x$

$\Rightarrow u - 1/n < u \leq x$

$\Rightarrow u \leq \sup S$ — (ii)

from (i) & (ii) $u = \sup S$

Existance of $\sqrt{2}$

Theorem :- there exist a real number x such that $x^2 = 2$

Proof :-

Consider $S = \{t \in \mathbb{R} ; 0 \leq t^2 < 2\}$

here $S \neq \Phi$ because $1 \in S$

also S is bounded above by 2

$\Rightarrow$ By completeness property, S has sup say x

~~also by Trichotomy law~~

also either $x^2 = 2$, or $x^2 < 2$ or $x^2 > 2$

Case (i) let $x^2 < 2$

we will find n

$\Rightarrow n \in \mathbb{N}$ such that $x + \frac{1}{n} \in S$

~~which is co~~

consider $(x+\frac{1}{n})^2 = x^2 + \frac{2x}{n} + \frac{1}{n^2}$

$< x^2 + \frac{2x}{n} + \frac{1}{n} \quad \left\{ \because \frac{1}{n^2} < \frac{1}{n} \right\}$

$= x^2 + \left(\frac{2x+1}{n}\right)$ — (1)

Now we also choose n such that

$$\frac{2x+1}{n} < 2 - x^2 \quad \text{— (ii)}$$

Such n is possible by archimedean property

$\therefore$ (2) $\Rightarrow \frac{1}{n} < \frac{2-x^2}{2x+1}$

$\left\{ \frac{2x+1}{2-x^2} < n \right.$

because by assumption $2-x^2 > 0 \Rightarrow \frac{2-x^2}{2x+1} > 0$
$\Rightarrow$ by 2nd cor $\frac{1}{n} < \frac{2-x^2}{2x+1}$

Now $2-x^2 > 0 \Rightarrow \frac{2-x^2}{2x+1} > 0$

(1) $\Rightarrow \left(x+\frac{1}{n}\right)^2 < x^2 + 2 - x^2$
$= 2$

which is contradiction because ~~$x+\frac{1}{n} \in S$~~ as x is upper bound of S

$\therefore x^2 \not< 2$

Case (ii) let $x^2 > 2$

~~$\Rightarrow$~~ consider $\left(x-\frac{1}{m}\right)^2 = x^2 - \frac{2x}{m} + \frac{1}{m^2}$
$> x^2 - \frac{2x}{m}$ — (3)

we will choose m such that $\left\{ \frac{2x}{x^2-2} < m \right\}$

$$\frac{2x}{m} < x^2 - 2 \quad \text{— (4)}$$

$\Rightarrow \frac{x^2-2}{2x} > \frac{1}{m}$

~~by 3~~ (3) $\Rightarrow \left(x-\frac{1}{m}\right)^2 > 2$

$\Rightarrow$ if $s \in S$
$\Rightarrow s^2 < 2 < \left(x-\frac{1}{m}\right)^2$

$\Rightarrow x - \frac{1}{m}$ is an upper bound of S
which is contradiction

$\Rightarrow x^2 \not> 2$

$\Rightarrow x^2 = 2$

Density of Rational number in $\mathbb{R}$

Density theorem :- If x & y are real numbers with $x<y$, then $\exists$ a rational number $r \in \mathbb{Q}$ such that $x<r<y$.

(any)

Proof :- Without loss of generality let $x>0$

$\because x<y \Rightarrow y-x>0$

$\Rightarrow \exists\, n \in \mathbb{N}$ such that $1/n < y-x$

∵ for $t>0$ $\exists\, n_t \in \mathbb{N}$ s.t. $0<1/n<t$ $\Rightarrow$ cor ③ of archimedean property

$\Rightarrow nx+1<ny$ — ①

as $nx>0$

$\exists\, m \in \mathbb{N}$ such that $m-1 \le nx < m$ — ②

{ for $y>0$ $\exists\, n_y \in \mathbb{N}$ such that $n_y - 1 \le y < n_y$

$\Rightarrow m \le nx+1 < m+1$

~~for $y>0$ $\exists\, n_y$~~

from ① & ⑪

$m \le nx+1 < ny$

also $nx<m<ny$

∵ $nx<m$ in ② and $nx+1<ny$ in ① also $m \le nx+1$ $\Rightarrow nx<m\le nx+1 < y$

$\Rightarrow x<\frac{m}{n}<y$

$\Rightarrow r=\frac{m}{n} \in \mathbb{Q}$ such that $x<r<y$

Corollary :- If x and y are real numbers with $x<y$, then there exist an irrational number z such that $x<z<y$.

Proof :- $\because x<y$

$\Rightarrow \frac{x}{\sqrt{2}} < \frac{y}{\sqrt{2}}$

{ ∵ $a<b$ & $c>0$ $\Rightarrow ac<bc$

By density theorem $\exists\, r \in \mathbb{Q}$ such that

$$\frac{x}{\sqrt{2}} < r < y/\sqrt{2}$$

$$\Rightarrow \quad x < r\sqrt{2} < y$$

$\ominus$ Here $r\sqrt{2}$ is an irrational no.

Qu (6) let A & B be bounded nonempty subsets of $\mathbb{R}$ let $A+B = \{a+b \mid a \in A, b \in B\}$. Prove that
$\sup(A+B) = \sup A + \sup B$ and
$\inf(A+B) = \inf A + \inf B$.

Sol^n :- $\because$ A & B are Bounded sets

$\Rightarrow$ inf, sup exist

let $\sup A = u_1$, $\inf A = l_1$,
$\sup B = u_2$, $\inf B = l_2$

$\Rightarrow l_1 \le a \le u_1 \quad \forall a \in A$

& $l_2 \le b \le u_2 \quad \forall b \in B$

To Prove :- $\sup(A+B) = \sup A + \sup B$
$\inf(A+B) = \inf A + \inf B$

$\because$ $A+B = \{a+b;\ a \in A, b \in B\}$

let $\sup(A+B) = u_3$

$\because$ $a \le u_1$ $\quad \forall a \in A, b \in B$
$b \le u_2$

$\Rightarrow$ $a+b \le u_1 + u_2$

$\Rightarrow$ $u_1 + u_2$ is upper bound of $A+B$

$\Rightarrow \sup(A+B) \le u_1 + u_2$

$\Rightarrow \sup(A+B) \le \sup(A) + \sup B$ — (I)

$\Rightarrow$ $\sup(A+B) \le$ upper bound of $A+B$

let J be another upper bound of $A+B$

$a+b \le J \quad \forall a+b \in A+B$

$\Rightarrow a \le J - b \quad \forall a \in A$

$\Rightarrow$ $J-b$ is upper bound of A

$\Rightarrow u_1 \le J - b$

$\Rightarrow J - u_1 \ge b \quad \forall b \in B$

$\Rightarrow$ $J - u_1$ is upper bound of B

$\Rightarrow u_2 \le J - u_1$

$\Rightarrow u_1 + u_2 \le J \Rightarrow u_1 + u_2 \le \sup(A+B)$

$\Rightarrow \sup(A) + \sup B \le \sup(A+B)$ — (II)

from (I) & (II) $\Rightarrow \sup(A+B) = \sup A + \sup B$.